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Tracer Dynamics in Dyson's Model of Interacting Brownian Particles

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We prove that the mean square displacement of a tracer particle grows as $\log t$ for large t. We point out a connection to the low-temperature floating phase of the ANNNI model.

KEY WORDS: Mean square displacement; bulk fluctuations; current across the origin.

1. BROWNIAN PARTICLES INTERACTING THROUGH A $1/x$ FORCE

We consider Brownian particles in one dimension interacting through the pair force $1/x$. The equations of motion are

$$dx_j(t) = \sum_{i, i \neq j} \frac{1}{x_j(t) - x_i(t)} dt + db_j(t) \tag{1.1}$$

with label $j = 0, \pm 1, \ldots$. Here $x_j(t) \in \mathbb{R}$ is the position of the jth particle at time t, and $\{b_j(t), j \in \mathbb{Z}\}$ are a collection of independent standard Brownian motions. The pair force between particles is repulsive. The corresponding pair potential is $-\log x$. The particles cannot cross with probability one and their order is preserved in the course of time.

Actually, the noncrossing is somewhat subtle. This can be seen already for two particles. Let us denote by $r_t \geq 0$ their relative distance. By (1.1) it satisfies

$$dr_t = \frac{1}{r_t} dt + dW_t \tag{1.2}$$

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This is the Bessel process, i.e., the radial part of the two-dimensional Brownian motion. It misses the origin with probability one, which implies that r_t never reaches zero.

For N particles we have to use Dyson's matrix representation[8]: We consider $N \times N$ complex, symmetric matrices. They are determined by N^2 real parameters. The matrix elements fluctuate according to independent Ornstein–Uhlenbeck processes, the diagonal elements twice as fast as the off-diagonal ones. This defines an N^2-dimensional, nondegenerate diffusion process. The N eigenvalues of this fluctuating matrix satisfy (1.1) with a confining, harmonic external potential. The coincidence of at least two eigenvalues of a complex, symmetric matrix defines a surface of codimension three in parameter space, which with probability one is never reached by the diffusion process. Therefore, given that initially all eigenvalues are distinct, they never cross.

In the following I completely suppress the limit procedure: expectations are defined for finite N first with the subsequent limit $N \to \infty$.

In Ref. 1, I investigated bulk properties of the system, in particular the fluctuations in the density on a large space-time scale (hydrodynamic limit). Having presented my results at the conference at Trebon, H. van Beijeren asked about the motion of a tracer particle in this system. To be specific: How does

$$\langle [x_j(t) - x_j(0)]^2 \rangle \tag{1.3}$$

behave for large t? Here the average is in the stationary process for (1.1) with density ρ. Of course, more refined information, such as

$$\langle \exp\{ik[x_j(t) - x_j(0)]\} \rangle \tag{1.3$'$}$$

is also welcome.

From the point of view of the technique used in Ref. 1, this appears to be a difficult problem: only functions symmetric in the label of particles map onto fermionic operators. D. Dürr and S. Goldstein explained to me that, since particles cannot cross, the position of the tracer particle is simply related to the current across the origin, which is a bulk quantity. Exploiting this idea, we prove that

$$\langle [x_j(t) - x_j(0)]^2 \rangle \cong (\pi\rho)^{-2} \log t \tag{1.4}$$

for large t.

Let us consider for a moment a general system of interacting Brownian particles in one dimension with pair force $F(x) = -V'(x)$, i.e., let us replace in (1.1) $1/x$ by $F(x)$.

If V is bounded, of finite range, and superstable, then

$$\langle [x_j(t) - x_j(0)]^2 \rangle \cong Dt \tag{1.5}$$

for large t with $0 < D_0 \leqslant D \leqslant 1$. The lower bound D_0 is the diffusion coefficient of a tracer particle in the static environment,

$$dx_t = -\sum_j V'(x_t - x_j)\, dt + db(t) \tag{1.6}$$

The $\{x_j\}$ are distributed according to the Gibbs measure for V with density ρ. Explicitly, because of one dimension, $D_0 = (\langle e^{-2V(0)} \rangle \langle e^{2V(0)} \rangle)^{-1}$, $V(x) = \sum V(x - x_j)$. In fact, $\varepsilon x_j(\varepsilon^{-2} t)$ converges to Brownian motion as $\varepsilon \to 0$.[2-4]

If V is still of finite range, but diverges sufficiently fast at the origin such that particles cannot cross with probability one, then

$$\langle [x_j(t) - x_j(0)]^2 \rangle \cong \frac{1}{\rho^2} D t^{1/2} \tag{1.7}$$

for large t. The noncrossing of particles reduces the fluctuations in the motion of a tracer particle. Relation (1.7) is a consequence of Ref. 5 under the extra hypothesis $V \geqslant 0$. This will be discussed in the Remark at the end of Section 2.

I conclude that the long-range repulsive part of the $1/x$ force even further reduces the fluctuations in the motion of a tracer particle. The $1/x$ force makes the system of Brownian particles very rigid.

The intermediate cases are not understood.

2. MEAN SQUARE DISPLACEMENT OF A TRACER PARTICLE

We pick as tracer particle the first particle to the right of the origin at time $t = 0$. We denote its position by x_t and want to study $\langle (x_t - x_0)^2 \rangle$. Averages are always in the stationary process with density ρ.

Let $J(t)$ be the current across the origin integrated over the time interval $[0, t]$, $t \geqslant 0$,

$J(t)$ = number of particles that cross 0 from left to right
− number of particles that cross 0 from right to left
during time $[0, t]$

Let $n(x)$ be the number of particles in the interval $[0, x]$,

$$n(x) = \begin{cases} \text{number of particles in } [0, x], & x \geqslant 0, \\ -\text{ number of particles in } [x, 0], & x < 0 \end{cases}$$

at time $t = 0$. The number n is increasing and takes only integer values.

We denote by f^{-1} the inverse function; $f^{-1} \circ f(x) = x$. In particular, n^{-1} is the inverse of n that is continuous to the left (lower semicontinuous). A simple geometric observation together with the time-reversibility of the process (1.1) yields

$$x_t = n^{-1}(J(t)) \tag{2.1}$$

Assuming for a moment that $|J(t)| \to \infty$ as $t \to \infty$, the ergodic theorem suggests

$$x_t \cong \frac{1}{\rho} J(t) \tag{2.2}$$

and therefore

$$\langle (x_t - x_0)^2 \rangle \cong \frac{1}{\rho^2} \langle J(t)^2 \rangle \tag{2.3}$$

To prove (1.4) we have to establish two properties. (i) $\langle J(t)^2 \rangle \cong \pi^{-2} \log t$. (ii) $\rho n^{-1}(J(t)) \cong J(t)$ for large t. Since $n(x)$ and $J(t)$ are not independent, this requires a little bit of work.

Lemma 1. We have

$$\lim_{t \to \infty} \frac{1}{\log t} \langle J(t)^2 \rangle = \pi^{-2} \tag{2.4}$$

Proof. Let

$$n(f, t) = \sum_j f(x_j(t)) \tag{2.5}$$

and let $J(f, t) = \int dx\, f(x)\, J(x, t)$, where $J(x, t)$ is the current across x integrated over the time interval $[0, t]$. Then by the conservation of the number of particles

$$n(f, t) - n(f, 0) = J(f', t) \tag{2.6}$$

Therefore

$$\begin{aligned} 2 \int dk\, \hat{f}(k)\, \hat{g}(k) [S(k, 0) - S(k, t)] \\ = \langle [n(f, t) - n(f, 0)][n(g, t) - n(g, 0)] \rangle \\ = -\langle J(f'', t)\, J(g, t) \rangle \end{aligned} \tag{2.7}$$

Here the first identity is the definition of the dynamic structure function $S(k, t)$,[1] and the second identity uses translation invariance. We choose a sequence of test functions f' and g tending to $\delta(x)$. This yields

$$\langle J(t)^2\rangle = \frac{1}{\pi}\int dk\, k^{-2}[S(k, 0) - S(k, t)] \tag{2.8}$$

From Ref. 1,

$$S(k, t) = (2\pi\, |k|\, t)^{-1}\{\exp[-t(k^2 + 2\pi\rho\, |k|)/2] - \exp(-t\, |k^2 - 2\pi\rho\, |k||/2)\} \tag{2.9}$$

The leading contribution to (2.8) is

$$\pi^{-2}\int_0^{2\pi\rho t} dk\, k^{-1}(1 - e^{-\pi\rho k}) \cong \pi^{-2}\log(2\pi\rho t) \tag{2.10}$$

for large t. ∎

We now want to show (2.3). Let us fix $L > 0$, eventually to be large. We define the functions

$$g_{+j}(x) = Lj + \rho x + \sqrt{x}, \qquad x \geqslant 0$$

$$g_{-j}(x) = \begin{cases} 0, & 0 \leqslant x \leqslant L(j+1) \\ \rho[x - L(j+1)] + [x - L(j+1)]^{1/2}, & L(j+1) \leqslant x \end{cases}$$

$$g_{\pm j}(-x) = -g_{\pm j}(x) \tag{2.11}$$

$j = 0, 1,\ldots$. Actually, because the system is so rigid, $x^{1/2}$ could be replaced by x^γ with any $0 < \gamma < 1$.

Let $\mathfrak{X}$ be the space of locally finite configurations over $\mathbb{R}$. We define the subsets $A_j \subset \mathfrak{X}$ by

$$\begin{aligned} A_j &= \{g_{+j-1} < n \leqslant g_{+j}\} \\ A_0 &= \{g_{-0} < n \leqslant g_{+0}\} \\ A_{-j} &= \{g_{-j} < n \leqslant g_{-j-1}\} \end{aligned} \tag{2.12}$$

$j = 1, 2,\ldots$. We let $\chi(A)$ denote the indicator function of the set A.

Using (2.1), we obtain the bounds

$$\begin{aligned}
&\langle g_{+0}^{-1}(J(t))^2\,\chi(A_0)\rangle + \sum_{j=1}^{\infty} \langle g_{+j}^{-1}(J(t))^2\,\chi(A_j)\rangle \\
&\quad + \sum_{j=1}^{\infty} \langle g_{-j-1}^{-1}(J(t))^2\,\chi(A_{-j})\rangle \\
&\quad \leqslant \langle x_t^2\rangle \\
&\quad = \sum_{j=-\infty}^{\infty} \langle x_t^2\chi(A_j)\rangle \\
&\quad \leqslant \langle g_{-0}^{-1}(J(t))^2\,\chi(A_0)\rangle + \sum_{j=1}^{\infty} \langle g_{+j-1}^{-1}(J(t))^2\,\chi(A_j)\rangle \\
&\quad + \sum_{j=1}^{\infty} \langle g_{-j}^{-1}(J(t))^2\,\chi(A_{-j})\rangle
\end{aligned} \tag{2.13}$$

We indicate how to estimate the contribution of the various terms. By Hölder's inequality we decouple the average over $J(t)$ and $\chi(A_j)$. This results in terms of the form $\text{const}\cdot\langle J(t)^2\rangle^{\alpha}$, $\alpha=0$, 1/4, 1/2, 3/4. The coefficients are either bounded or can be bounded by

$$\sum_{j=1}^{\infty} (Lj)^2\langle\chi(\{n(x)=0 \text{ for } |x|\leqslant (j+1)\,L\})\rangle \tag{2.14}$$

In Lemma 2 below, we show that the sum is finite. The only error term that grows as $\log t$ is

$$\langle J(t)^2[1-\chi(A_0)]\rangle \leqslant \langle J(t)^4\rangle^{1/2}\langle[1-\chi(A_0)]\rangle^{1/2} \tag{2.15}$$

By a computation similar to the one in the proof of Lemma 1, only more lengthy, one establishes that $\langle J(t)^4\rangle$ grows as $(\log t)^2$. Therefore, dividing in (2.13) by $\log t$ yields

$$\begin{aligned}
&(\rho\pi)^{-2} - c\langle 1-\chi(A_0)\rangle^{1/2} \\
&\quad \leqslant \lim_{t\to\infty} \frac{1}{\log t}\langle x_t^2\rangle \\
&\quad \leqslant (\rho\pi)^{-2} + c\langle 1-\chi(A_0)\rangle^{1/2}
\end{aligned} \tag{2.16}$$

We still have to show that for L large enough: (iii) the probability to have $[0, L]$ free of particles is small; and (iv) $\mathfrak{X}\backslash A_0$ has a small measure.

Lemma 2. We have

$$\langle\chi(\{n(x)=0 \text{ for } 0\leqslant x\leqslant L\})\rangle \leqslant e^{-\rho L} \tag{2.17}$$

Proof. The correlation functions of the equilibrium measure, $t=0$, are of determinantal form.[1] Therefore

$$\begin{aligned}\langle\chi(\{n(x)=0 \text{ for } 0\leqslant x\leqslant L\})\rangle \\ &= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\int_0^L dx_1\cdots\int_0^L dx_m \det R(x_i-x_j)\Big|_{i,j=1,\dots,m} \\ &= \det(1-Q_L R Q_L)\end{aligned} \tag{2.18}$$

as a Fredholm determinant. Here Q_L and R are projection operators acting in the Hilbert space $L^2(\mathbb{R}, dx)$. The operator Q_L projects into the interval $[0, L]$ and R projects onto the interval $[-\pi\rho, \pi\rho]$ in momentum space, i.e., as an integral kernel

$$R(x-y)=\frac{1}{2\pi}\int_{-\pi\rho}^{\pi\rho} dk\, e^{ik(x-y)} \tag{2.19}$$

Now, tr $Q_L R=\rho L$, and therefore

$$\det(1-Q_L R Q_L)\leqslant \exp(-\text{tr}\, Q_L R)=\exp(-\rho L) \quad ▮ \tag{2.20}$$

Lemma 3. We have for $x\geqslant x_0>0$

$$\langle\chi(\{n(x)-\rho x\geqslant\sqrt{x}\})\rangle\leqslant c\exp(-x^{1/4}) \tag{2.21}$$

Proof. We use the exponential Chebyshev inequality,

$$\begin{aligned}&\langle\chi(\{n(x)-\rho x\geqslant\sqrt{x}\})\rangle \\ &\qquad\leqslant\exp(-z-z/\sqrt{x})\langle\exp[zn(x)/\sqrt{x}]\rangle\end{aligned} \tag{2.22}$$

Now

$$\langle e^{zn(x)}\rangle=\det[1+RQ_x(e^z-1)]\leqslant\exp[\rho x(e^z-1)] \tag{2.23}$$

By choosing $z=x^{1/4}$, we obtain (2.21). ▮

Lemma 4. We have for $x_0>0$

$$\langle\chi(\{n(x_0)\geqslant L\})\rangle\leqslant c_1 e^{-c_2 L} \tag{2.24}$$

Proof. As the previous one.

From Lemmas 2–4 together with the Borel–Cantelli Lemma we conclude that $\langle 1-\chi(A_0)\rangle$ tends to zero as $L\to\infty$.

Proposition. For the stationary process (1.1) with density ρ we have

$$\lim_{t\to\infty} \frac{1}{\log t} \langle [x_j(t) - x_j(0)]^2 \rangle = (\pi\rho)^{-2} \tag{2.25}$$

It is not known whether $(\log t)^{-1/2} J(t)$ has a limiting distribution as $t \to \infty$, in particular, whether it tends to a Gaussian.

Remark. Using the same technique, we want to consider the case of interacting Brownian particles with a finite-range potential, which, however, diverges sufficiently fast at the origin such that particles cannot cross. We first observe that

$$\langle [n(f, t) - n(f, 0)]^2 \rangle = 2 \int dk\, |\hat{f}(k)|^2 [S(k, 0) - S(k, t)] \tag{2.26}$$

and therefore

$$0 \leqslant S(k, 0) - S(k, t) \tag{2.27}$$

Let $n(f) = n(f, 0)$ and let L be the generator for the interacting Brownian particles. By the spectral theorem and Jensen's inequality,

$$\begin{aligned} &\int dk\, |\hat{f}(k)|^2\, S(k, t) \\ &\quad = \langle [n(f) - \langle n(f) \rangle]\, e^{Lt} [n(f) - \langle n(f) \rangle] \rangle \\ &\quad \geqslant \langle [n(f) - \langle n(f) \rangle]^2 \rangle \\ &\qquad \times \exp\{ -t \langle n(f)\, \mathrm{l}n(f) \rangle / \langle [n(f) - \langle n(f) \rangle]^2 \rangle \} \\ &\quad = \left[\int dk\, |\hat{f}(k)|^2\, S(k, 0) \right] \\ &\qquad \times \exp\left\{ -t\rho \int dk\, |\hat{f}(k)|^2\, k^2 \Big/ 2 \int dk\, |\hat{f}(k)|^2\, S(k, 0) \right\} \end{aligned} \tag{2.28}$$

Therefore

$$S(k, 0) - S(k, t) \leqslant S(k, 0) \{ 1 - \exp[-t\rho k^2 / 2S(k)] \} \tag{2.29}$$

Now

$$\frac{1}{\sqrt{t}} \langle J(t)^2 \rangle = \frac{1}{\pi} \int dk\, k^{-2} [S(k/\sqrt{t}, 0) - S(k/\sqrt{t}, t)] \tag{2.30}$$

For positive potentials of finite range we prove in Ref. 5 that pointwise

$$\lim_{t \to \infty} S(k/\sqrt{t}, t) = \chi \exp(-k^2\rho/2\chi) \tag{2.31}$$

with $\chi = S(0, 0)$ the static compressibility. Relation (2.29) provides a uniform upper bound for the integrand in (2.30). Therefore

$$\lim_{t \to \infty} \frac{1}{\sqrt{t}} \langle J(t)^2 \rangle = (\rho\chi/2\pi)^{1/2} \tag{2.32}$$

Estimates such as those of Lemmas 2–4 are standard for Gibbs measures. We conclude that, if $V \geqslant 0$, $V(-x) = V(x)$, V three times differentiable, $\lim_{x \to 0} V(x)/(\log |x|) = \infty$, and V of finite range, then

$$\lim_{t \to \infty} \frac{1}{\sqrt{t}} \langle [x_j(t) - x_j(0)]^2 \rangle = \rho^{-2}(\rho\chi/2\pi)^{1/2} \tag{2.33}$$

I expect that the method of Ref. 5 would also allow one to show that $\varepsilon^{1/2}J(\varepsilon^{-2}t)$ tends to a Gaussian process as $\varepsilon \to 0$.

3. SPIN–SPIN CORRELATIONS OF THE FLOATING PHASE OF THE ANNNI MODEL

At a specific value of the couplings the two-dimensional ANNNI model has a zero-temperature phase of the form

$$\begin{matrix} + + + + - - - + + - - - - - - + \\ \cdots + + + + - - - + + - - - - - - + \cdots \\ + + + + - - - + + - - - - - - + \end{matrix}$$

The length of intervals with a given sign is arbitrary, but larger than or equal to two. As we increase the temperature slightly, a natural approximation is to describe the $+-$ interfaces as random walks with the constraint that their distance is always larger than or equal to two.[6] As explained in Ref. 1 (cf. also Ref. 7), interacting Brownian motion governed by (1.1) is the continuum version of this model.

Let us label the particles in (1.1) in their natural order as

$$\cdots x_{-1}(0) \leqslant 0 < x_0(0) < x_1(0) < \cdots \tag{3.1}$$

We define the spin field $\sigma(x, t)$ by

$$\sigma(x, t) = \begin{cases} 1, & \text{if } x_j(t) \leqslant x < x_{j+1}(t) \text{ and } j \text{ odd} \\ -1, & \text{if } x_j(t) \leqslant x < x_{j+1}(t) \text{ and } j \text{ even} \end{cases}$$

From the point of view of the ANNNI model the object of interest is the spin–spin correlation

$$\langle \sigma(x, t)\, \sigma(0, 0) \rangle \tag{3.2}$$

average in the stationary process (1.1) with density ρ.

The same geometric observation as in the case of the tracer particle yields

$$\sigma(x, t) = \exp\{i\pi[n(x) + J(t)]\} \tag{3.3}$$

Since $\sigma(0, 0) = 1$ by construction,

$$\langle \sigma(x, t)\, \sigma(0, 0) \rangle = \langle \exp\{i\pi[n(x) + J(t)]\} \rangle \tag{3.4}$$

In the physical literature (Ref. 6 and references therein), this expectation is computed assuming that $n(x)$ for large x and $J(t)$ for large t have a Gaussian distribution.

We use the conservation law (2.6) to express $J(t)$ through the density field. Going through the algebra in Ref. 1 then yields

$$\langle \sigma(x, t)\, \sigma(0, 0) \rangle = \det[1 + (Re^{-t\Delta/2} P_x e^{t\Delta/2} P_0 - R)] \tag{3.5}$$

for $t \geqslant 0$. Again R, Δ, and P_x are linear operators acting on the Hilbert space $L^2(\mathbb{R}, dy)$, and Δ is the Laplacian. Note that $Re^{-t\Delta/2}$ is a bounded operator. P_x is the multiplication operator

$$P_x f(y) = \begin{cases} -f(y) & \text{for} \quad y < x \\ f(y) & \text{for} \quad x < y \end{cases} \tag{3.6}$$

The operator in the parentheses is of trace class.

I have no idea how to estimate the large-x and -t behavior of the determinant in (3.5). My only observation is that for $t = 0$ we have

$$\langle \sigma(x, 0)\, \sigma(0, 0) \rangle = \det(1 - 2Q_x R Q_x) \tag{3.7}$$

with Q_x defined below (2.18). Since $\|RQ_x\| \leqslant 1$ and since $Q_x R Q_x$ increases to R as $x \to \infty$, $2Q_x R Q_x$ must have eigenvalues that cross 1 for increasing x. Therefore, the spin–spin correlation oscillates around zero. Using $\log |1 - 2u| \leqslant -2u(1-u)$ for $0 \leqslant u \leqslant 1$, we have the upper bound

$$\begin{aligned} &|\langle \sigma(x, 0)\, \sigma(0, 0) \rangle| \\ &\quad \leqslant \exp[-2 \operatorname{tr} Q_x R + 2 \operatorname{tr}(Q_x R)^2] \\ &\quad \cong |x|^{-\rho} \end{aligned} \tag{3.8}$$

for large $|x|$. This is believed to be the correct asymptotic behavior.

Remark. Yet another physical realization of (1.1) is the terrace step kink model.[9,10] It describes a two-dimensional stepped surface of a crystal. At low temperatures the steps are of height one and well separated from each other. If the surface is slightly tilted against the z axis and if the steps run essentially parallel to the y axis, then their locations are given by the lines $y \to u_m(y)$, $y \in \mathbb{Z}$, $m = 1, 2,\ldots$. The energy of a collection of steps is

$$H = \sum_m \sum_y [u_m(y+1) - u_m(y)]^2 + \sum_m \sum_y g(u_{m+1}(y) - u_m(y)) \qquad (3.9)$$

For $g(0) = \infty$, $g(u) = 0$ for $u > 0$, this is essentially equivalent to (1.1). Physically, g also has an attractive part. Mapped onto a dynamical model of interacting Brownian particles, they no longer have pair forces.

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